Logic, Probability and Indicative Conditionals

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Readings

Suggested:

- E. Adams, *The Logic of Conditionals* (1975), esp. Ch. 1–3.
- E. Adams, *A Primer of Probability Logics* (1996)
- Lewis, D. (1976). Probabilities of conditionals and conditional probabilities. In *IFS: Conditionals, Belief, Decision, Chance and Time* (pp. 129–147).

Plan

1. Indicative Conditionals
2. Probability and Logic
3. Conditional Probability
4. Lewis Triviality
5. Counterfactuals

Outline

Warm-up Puzzle

"If it is sunny, I bike to work."

Over the last 100 workdays, you kept stats:

	went by bike (q)	didn't go by bike ($\neg q$)
sunny (p)	8	12
not sunny ($\neg p$)	7	73

How 'true' is your claim "*If it is sunny, I bike to work.*"?

Intuitively, you look only at the days when *it was sunny*:
$P(q \mid p) = \frac{8}{8+12} = \frac{8}{20} = 40\%.$

If we identified it with the material conditional $p \supset q \;\equiv\; \neg p \vee q$, the latter is false only on days with $p \wedge \neg q$ (sunny but didn't go by bike), i.e. 12 out of 100 days. So $P(\neg p \vee q) = 1 - P(p \wedge \neg q) = 1 - \frac{12}{100} = 88\%.$

But it is absurd. Our intuitive evaluation tracks $P(q \mid p)$, not the material conditional $\neg p \vee q$.

Indicative vs. Subjunctive (Counterfactual) Conditionals

- **Subjunctive / counterfactual** conditionals: usually about *non-actual* or *contrary-to-fact* situations.
- **Indicative** conditionals: usually about *epistemic uncertainty* about the actual world.

(1) a. If Oswald hadn't shot Kennedy, then someone else would have.
    b. $\varphi \rightsquigarrow \psi$ (counterfactual)

(2) a. If Oswald didn't shoot Kennedy, then someone else did.
    b. $\varphi \rightarrow \psi$ (indicative)

- Counterfactuals: "What would have happened, if . . . " (non-truth-functional, possible-worlds semantics).
- Indicatives: "Given what we know, if . . . , then . . . " (good candidates for a probabilistic treatment).

Indicative Conditionals and Material Implication

From classical logic:

$$\varphi \to \psi \quad \overset{?}{\equiv} \quad \varphi \supset \psi \quad \equiv \quad \neg\varphi \lor \psi \quad \equiv \quad \neg(\varphi \land \neg\psi).$$

We do seem to accept some inferences that look material:

Or-to-if:

$$\varphi \lor \psi \models \neg\varphi \to \psi$$

Not-and-to-if:

$$\neg(\varphi \land \psi) \models \varphi \to \neg\psi$$

Moreover, to preserve **modus ponens** for indicatives:

$\to$**-to-**$\supset$**:** $\varphi \to \psi \models \varphi \supset \psi$

For Identifying Indicative with Material Conditional

Ad absurdum conditionals are compelling:

(3) If you can run 100 km without stopping, I will eat my hat.

We take this as a good way to argue:

- We are confident that I will not eat my hat.
- So (by contrapositive reasoning) we conclude that you cannot run 100 km without stopping.

Material implication validates such reasoning patterns naturally.

Gibbard's Collapse Theorem

(P1) $\varphi \rightarrow \psi \models \varphi \supset \psi$ (assumption)

(P2) If $\varphi \models \psi,$ then $\models \varphi \rightarrow \psi$ (Conditional Proof)

(P3) $\varphi \rightarrow (\psi \rightarrow \chi) \equiv (\varphi \wedge \psi) \rightarrow \chi$ (Import-Export)

(1) $(\varphi \supset \psi) \rightarrow (\varphi \rightarrow \psi) \equiv ((\varphi \supset \psi) \wedge \varphi) \rightarrow \psi$ (instance of (P3))

(2) $((\varphi \supset \psi) \wedge \varphi) \rightarrow \psi$ (since $(\varphi \supset \psi) \wedge \varphi \models \psi$ and (P2))

(3) $(\varphi \supset \psi) \rightarrow (\varphi \rightarrow \psi)$ (by (1) and (2))

(4) $(\varphi \supset \psi) \rightarrow (\varphi \rightarrow \psi) \models (\varphi \supset \psi) \supset (\varphi \rightarrow \psi)$ (by (P1))

(5) $(\varphi \supset \psi) \supset (\varphi \rightarrow \psi)$ (by (3) and (4))

(6) $\varphi \supset \psi \models \varphi \rightarrow \psi$ (by Deduction Theorem)

From (P1) and (6) we obtain mutual entailment between $\varphi \rightarrow \psi$ and $\varphi \supset \psi$, hence equivalence.

Against Indicative = Material Conditional

Let $q =$ "It rains", $p =$ "The proof is wrong".

Classically:

$$q \models p \supset q \qquad \text{and} \qquad \neg p \models p \supset q$$

So whenever it is raining, the material conditional $p \supset q$ is true, and whenever the proof is correct, $p \supset q$ is also true.

But in ordinary language:

- If the proof is wrong, it is raining.
- If the proof is not wrong, it is raining.

sound unjustified or bizarre in most contexts. Material implication makes any true consequent or any false antecedent make the conditional true.

Strict Implication

Another idea is to analyze indicative conditionals as *strict* implications:

$\Box(\varphi \supset \psi)$ or more explicitly: $\forall w \in W:$ if $w \models \varphi,$ then $w \models \psi$.

But this does not remove the paradoxes:

$$\Box q \models \Box(p \supset q) \qquad \Box\neg p \models \Box(p \supset q)$$

So we get *paradoxes of strict implication* parallel to the material ones: if q is necessary, then necessarily “if p, then q”, etc.

Outline

1. Indicative Conditionals

2. Probability and Logic

3. Conditional Probability

4. Lewis Triviality

5. Counterfactuals

Why Bring in Probability?

Two roles for conditionals:

- **Truth-conditions**: what makes a conditional true or false.
- **Reasoning** / **assertibility**: when it is rational to accept or assert a conditional.

Rather than asking "what makes *If* φ, ψ true?", we ask:

When is it rational to assert *If* φ, ψ?

Probabilities give a measure of *degree of belief*. So we look at a probability as a candidate for the *degree of acceptability* of the indicative conditional.

Events and Probability Spaces

A *probability space* is a triple $(\Omega, \mathcal{F}, \Pr)$ where:

- $\Omega \neq \emptyset$ is a *sample space* (set of possible outcomes).
- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a set of *events*, closed under
  - complements: if $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$;
  - finite unions: if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$.
- $\Pr : \mathcal{F} \to [0, 1]$ is a *probability measure* such that:
  1. $\Pr(\Omega) = 1$;
  2. if $A, B \in \mathcal{F}$ and $A \cap B = \emptyset$, then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Examples: Events and Their Probabilities

Example 1: Fair coin

- $\Omega = \{H, T\}$.
- $\mathcal{F} = \mathcal{P}(\Omega) = \{\emptyset, \{H\}, \{T\}, \Omega\}$.
- $\Pr(\{H\}) = \Pr(\{T\}) = \frac{1}{2}$.

Then, for instance:

$$\Pr(\Omega) = 1, \qquad \Pr(\emptyset) = 0, \qquad \Pr(\{H\} \cup \{T\}) = \frac{1}{2} + \frac{1}{2} = 1.$$

Example 2: Fair die

- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- $\mathcal{F} = \mathcal{P}(\Omega)$.
- $\Pr(\{\omega\}) = \frac{1}{6}$ for each $\omega \in \Omega$.

Let

$$E = \{\text{even}\} = \{2, 4, 6\}, \qquad P = \{\text{prime}\} = \{2, 3, 5\}.$$

Then

$$\Pr(E) = \frac{3}{6} = \frac{1}{2}, \quad \Pr(P) = \frac{3}{6} = \frac{1}{2}, \quad \Pr(E \cap P) = \Pr(\{2\}) = \frac{1}{6}.$$

Conditional Probability

Often we want the probability of an event *given* that another event has occurred.

Example (fair die):

- Let $E = \{\text{even}\} = \{2, 4, 6\}$.
- Let $P = \{\text{prime}\} = \{2, 3, 5\}$.

Suppose we learn that the outcome is even (E). What is the probability that it is also prime (P)?

$$P(P \mid E) = \frac{\text{number of outcomes in } P \cap E}{\text{number of outcomes in } E} = \frac{|\{2\}|}{|\{2, 4, 6\}|} = \frac{1}{3}$$

for $P(B) > 0$,

$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)}$$

Chain Rule for Probabilities

Two events:

$$P(A \wedge B) = P(A) \cdot P(B \mid A)$$

whenever $P(A) > 0$.

Three events:

$$P(A \wedge B \wedge C) = P(A) \cdot P(B \mid A) \cdot P(C \mid A \wedge B)$$

provided all the conditional probabilities are defined.

For events $A_1, \ldots, A_n$:

$$P(A_1 \wedge \cdots \wedge A_n) = P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \wedge \cdots \wedge A_{n-1})$$

From Events to Probabilities of Sentences

So far, probabilities live on *events* $A, B \in \mathcal{F}$ in a space $(\Omega, \mathcal{F}, \mathrm{Pr})$ (coins, dice, etc.).

In *probabilistic semantics*, we now want to talk about probabilities of *sentences* in a propositional language $\mathcal{L}$:

- Intuitively, each sentence $\varphi \in \mathcal{L}$ corresponds to an event:

$$[\varphi] \ := \ \{\omega \in \Omega : \omega \models \varphi\}$$

- Rather than carrying $(\Omega, \mathcal{F}, \mathrm{Pr})$ around explicitly, we *abstract away* from it and work directly with a *probability function on the language*

$$P : \mathcal{L} \to \mathbb{R},$$

which is required to behave like Pr on the associated events $[\varphi]$.

Probabilities: Axioms

Fix a propositional language $\mathcal{L}$ (closed under $\neg, \wedge, \vee$) and classical consequence $\models$.

Definition (Probability function)

A *probability function* P on a propositional language $\mathcal{L}$ is a function $P : \mathcal{L} \to \mathbb{R}$ such that, for all $\varphi, \psi \in \mathcal{L}$:

1. $P(\varphi) \geq 0$.
2. If $\models \varphi$ (i.e. φ is a tautology), then $P(\varphi) = 1$.
3. If $\models \neg(\varphi \wedge \psi)$ (i.e. φ and ψ are mutually exclusive), then

$$P(\varphi \vee \psi) = P(\varphi) + P(\psi).$$

- For every φ, $P(\varphi) \in [0, 1]$. (From (1) and (2) applied to $\varphi \vee \neg\varphi$.)
- If φ and ψ are logically equivalent ($\models \varphi \leftrightarrow \psi$), then $P(\varphi) = P(\psi)$.

Some Useful Facts

Complement: $P(\neg\varphi) = 1 - P(\varphi)$

By (2), $P(\varphi \vee \neg\varphi) = 1$. By (3), and the fact that φ and $\neg\varphi$ are mutually exclusive,

$$P(\varphi \vee \neg\varphi) = P(\varphi) + P(\neg\varphi).$$

So $1 = P(\varphi) + P(\neg\varphi)$, hence $P(\neg\varphi) = 1 - P(\varphi)$

Conjunction never more probable than its conjuncts:

$$P(\varphi \wedge \psi) \leq P(\varphi)$$

$$P(\varphi) = P((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)) = P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi)$$

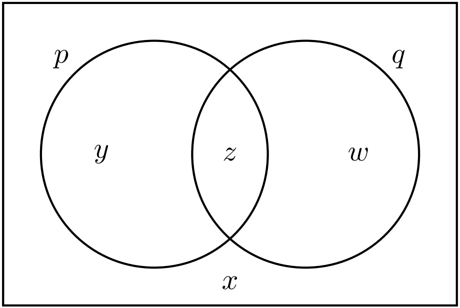
hence

$$P(\varphi \wedge \psi) = P(\varphi) - P(\varphi \wedge \neg\psi) \leq P(\varphi)$$

Disjunction (on blackboard): $P(\varphi \vee \psi) = P(\varphi) + P(\psi) - P(\varphi \wedge \psi)$

Venn Diagram Representation

It is often convenient to represent probabilities by means of Venn diagrams.



$$P(p) = y + z \qquad P(p \wedge q) = z \qquad P(p \vee q) = y + z + w$$

and $x + y + z + w = 1$.

Truth-Preserving Inference and Probability

Classical consequence:

$$\Gamma \models_{CL} \psi$$

In every classical valuation, if all sentences in Γ are true, then ψ is true.

Viewed probabilistically, this suggests:

If all premises in Γ are *certain*,
the conclusion should also be *certain*.

Formally, for any probability function P on $\mathcal{L}$:

$$(\forall \gamma \in \Gamma : \ P(\gamma) = 1) \implies P(\psi) = 1$$

But what if the premises do *not* have probability 1, only something close to 1? We want a probabilistic analogue of validity that tells us how much probability can be "lost" from premises in Γ to a conclusion ψ.

Probabilities Can Decrease Along Valid Inference

Consider the valid inference:

$$p \vee q,\ p \supset q \models_{CL} q$$

For an *arbitrary* probability function P, we can compute $P(q)$ in terms of the premises:

$$P(q) = P(p \vee q) + P(p \supset q) - 1$$

$$P(p \vee q) = P(q) + P(p \wedge \neg q),$$
$$P(p \supset q) = P(\neg(p \wedge \neg q)) = 1 - P(p \wedge \neg q)$$

Eliminating $P(p \wedge \neg q)$ yields

$$P(q) = P(p \vee q) + P(p \supset q) - 1$$

So in this special case the probabilities of the premises *determine* the probability of the conclusion.

Probabilities Can Decrease Along Valid Inference

In general, from a valid inference we usually get only **bounds** on $P(\psi)$ in terms of the premises.

For example, from $p \supset q,\ p \models_{CL} q$ one can show (**blackboard)**:

$$\text{If } P(p \supset q) = a \text{ and } P(p) = b, \text{ then } a + b - 1 \leq P(q) \leq a.$$

In particular, if both premises are very probable, $P(p \supset q) \geq 1 - \epsilon$ and $P(p) \geq 1 - \epsilon$, then $P(q) \geq 1 - 2\epsilon$.

Classical Entailment and Probability

For sentences φ, ψ the following are equivalent:

1. $\varphi \models_{CL} \psi$.
2. For every probability function P, $P(\varphi) \leq P(\psi)$.
3. For every P and every $\epsilon \geq 0$: $P(\varphi) \geq 1 - \epsilon \;\Rightarrow\; P(\psi) \geq 1 - \epsilon$

Classical Entailment and Probability: (1 $\Rightarrow$ 2)

If $\varphi \models_{CL} \psi$, then $\varphi \wedge \neg\psi$ is classically inconsistent, so $P(\varphi \wedge \neg\psi) = 0.$

Hence

$$P(\varphi) = P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi) = P(\varphi \wedge \psi) \leq P(\psi).$$

So $P(\varphi) \leq P(\psi)$ for every probability function P.

Classical Entailment and Probability: (2 $\Rightarrow$ 3)

Assume (2): for every probability function P, $P(\varphi) \leq P(\psi)$.

Now let P be any probability function and $\epsilon \geq 0$ such that

$$P(\varphi) \geq 1 - \epsilon.$$

By (2), $P(\varphi) \leq P(\psi)$, so $P(\psi) \geq P(\varphi) \geq 1 - \epsilon$.

Thus

$$P(\varphi) \geq 1 - \epsilon \;\Rightarrow\; P(\psi) \geq 1 - \epsilon$$

for every P and every $\epsilon \geq 0$.

Classical Entailment and Probability: ($3 \Rightarrow 1$)

Assume (3). We argue by contraposition.

Suppose $\varphi \not\models_{CL} \psi$.

Then there is a valuation v with $v(\varphi) = 1$ and $v(\psi) = 0$.

(3) says something must hold for *every* probability function P and every $\epsilon \geq 0$. So to falsify (3), it is enough to build *one* specific P and *one* specific ϵ for which the implication fails.

We define a probability function P_v by:

$$P_v(\theta) = \begin{cases} 1 & \text{if } v(\theta) = 1, \\ 0 & \text{if } v(\theta) = 0. \end{cases}$$

Then $P_v(\varphi) = 1$ and $P_v(\psi) = 0$.

Take, for example, $\epsilon = \frac{1}{2}$. We have

$$P_v(\varphi) = 1 \geq 1 - \tfrac{1}{2}, \text{ but } P_v(\psi) = 0 \not\geq 1 - \tfrac{1}{2}.$$

From One Premise to Many Premises

For $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$,

$$\Gamma \models_{CL} \varphi \quad \text{iff} \quad \forall P\, \forall \epsilon \geq 0 : \ \forall i,\ P(\gamma_i) \geq 1 - \epsilon \ \Rightarrow \ P(\varphi) \geq 1 - n\epsilon$$

- Each premise may be false with probability at most ϵ.
- in the worst case, the "errors" in different premises do not overlap, so the total error in the conclusion can grow to at most $n\epsilon$.
- this uses only very weak probabilistic assumptions (no independence).[1]

[1] Under extra assumptions like independence, the bound can be sharpened (e.g. to $(1-\epsilon)^n$), but the general $1 - n\epsilon$ bound already characterizes classical consequence.

From Probability to Uncertainty

So far we have looked at how the probability of a conclusion can be *lower* than the probabilities of the premises in a valid inference.

It is often more convenient to track not "how probable" a sentence is, but "how much room there is for it to be *wrong*":

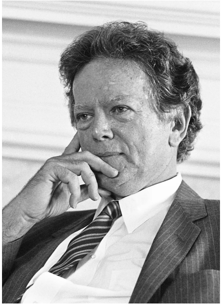
$$U_P(\varphi) := P(\neg\varphi) = 1 - P(\varphi)$$

- $U_P(\varphi)$ is the *risk of error* in accepting φ.
- If $P(\varphi)$ is close to 1, then $U_P(\varphi)$ is small: φ is very safe to rely on.
- If we reason from premises Γ to a conclusion ψ, we would like the uncertainty of ψ to be bounded by the *total* uncertainty we take on in accepting Γ.

This suggests the following probabilistic counterpart to truth-preserving validity:

In a good inference, the "error" in the conclusion should be no greater than the *sum* of the errors in the premises.

Probabilistic Entailment



Patrick Suppes (1922–2014)

Ernest W. Adams (1926–2009)

Probabilistic entailment $\models_P$

$$\Gamma \models_P \varphi \quad \text{iff} \quad \forall P : U_P(\varphi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$$

The uncertainty of the conclusion is no greater than the total uncertainty of the premises.

Adams notion

$$\Gamma \models_P \varphi \quad \text{iff} \quad \forall P:\ U_P(\varphi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$$

Adams-style:

$$\Gamma \models_a \varphi \quad \text{iff} \quad \forall \epsilon \geq 0,\ \exists \delta \geq 0 \text{ s.t.}$$

$$\forall P:\ (\forall \gamma \in \Gamma : P(\neg\gamma) \leq \delta) \Rightarrow P(\neg\varphi) \leq \epsilon$$

Whenever your premises are mostly correct within some tolerance δ, your conclusion will also be mostly correct within a tolerance ϵ.

Probabilistic Consequence (Adams, Theorem 3)

$$(a)\ \Gamma \models_{CL} \varphi \quad \text{iff} \quad (b)\ \Gamma \models_P \varphi \quad \text{iff} \quad (c)\ \Gamma \models_a \varphi$$

From Classical to P

Recall:

$$\Gamma \models_P \varphi \quad \text{iff} \quad \forall P: \ U_P(\varphi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma),$$

where $U_P(\alpha) := P(\neg\alpha)$.

Assume $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is finite and $\Gamma \not\models_P \varphi$. Then there is some P such that

$$U_P(\varphi) > \sum_{\gamma \in \Gamma} U_P(\gamma)$$

i.e.

$$P(\neg\varphi) > \sum_{i=1}^{n} P(\neg\gamma_i)$$

so $1 > P(\varphi) + \sum_{i=1}^{n} P(\neg\gamma_i)$

From Classical to P

$$\neg \bigwedge \Gamma \equiv \neg\gamma_1 \vee \cdots \vee \neg\gamma_n,$$

so by subadditivity

$$P(\neg \bigwedge \Gamma) \leq \sum_{i=1}^{n} P(\neg\gamma_i)$$

By subadditivity again,

$$P(\neg \bigwedge \Gamma \vee \varphi) \leq P(\neg \bigwedge \Gamma) + P(\varphi) \leq \sum_{i=1}^{n} P(\neg\gamma_i) + P(\varphi)$$

But $\bigwedge \Gamma \supset \varphi$ is classically equivalent to $\neg \bigwedge \Gamma \vee \varphi$, so

$P(\bigwedge \Gamma \supset \varphi) = P(\neg \bigwedge \Gamma \vee \varphi) \leq \sum_{i=1}^{n} P(\neg\gamma_i) + P(\varphi) < 1$

Thus $\bigwedge \Gamma \supset \varphi$ is not a tautology, so $\Gamma \not\models_{CL} \varphi$.

From P to Adams

Assume $\Gamma \models_P \varphi$ and let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$. Fix $\epsilon \geq 0$ and set

$$\delta := \frac{\epsilon}{n}$$

Now take any probability function P such that

$$P(\neg\gamma_i) \leq \delta \quad \text{for all } i = 1, \ldots, n$$

Then

$$P(\neg\varphi) = U_P(\varphi) \ \leq \ \sum_{i=1}^{n} U_P(\gamma_i) \ = \ \sum_{i=1}^{n} P(\neg\gamma_i) \ \leq \ n \cdot \delta \ = \ \epsilon$$

Since for every ϵ we have found such a δ, it follows that $\Gamma \models_a \varphi$.

From Adams Back to Classical

Assume $\Gamma \not\models_{CL} \varphi$. Then there is a valuation v with

$$v(\gamma) = 1 \text{ for all } \gamma \in \Gamma, v(\varphi) = 0$$

View v as a probability function P_v:

$$P_v(\theta) := \begin{cases} 1 & \text{if } v(\theta) = 1, \\ 0 & \text{if } v(\theta) = 0. \end{cases}$$

Then

$$P_v(\gamma) = 1 \text{ for all } \gamma \in \Gamma \quad P_v(\varphi) = 0$$

so

$$P_v(\neg\gamma) = 0 \text{ for all } \gamma \in \Gamma \quad P_v(\neg\varphi) = 1$$

From Adams Back to Classical

Fix, for example, $\epsilon = \frac{1}{2}$. For *any* $\delta \geq 0$ we have

$$\forall \gamma \in \Gamma,\ P_v(\neg\gamma) = 0 \leq \delta$$

but

$$P_v(\neg\varphi) = 1 > \epsilon$$

Thus for this ϵ there is *no* δ making the Adams condition true: the antecedent holds for P_v, while the consequent fails. Hence $\Gamma \not\models_a \varphi$.

Probabilistic Entailment and Classical Logic

- We have introduced several probabilistic consequence relations on a propositional language $\mathcal{L}$.
- **Adams' theorem:** All these probabilistic notions induce exactly *classical* consequence on $\mathcal{L}$.
- An inference is classically valid iff, for *every* probability function P, the *error* in the conclusion is never greater than the *total error* already accepted in the premises.
- In what follows we adopt the following as our default notion of probabilistic entailment:

$$\Gamma \models_P \varphi \quad \text{iff} \quad \forall P:\ U_P(\varphi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$$

Enriching the language with conditional probabilities

- When we enrich the language with expressions for conditional probabilities $P(\psi \mid \varphi)$, the induced consequence relation $\models_P$ is not equivalent to classical logic.
- But it validates exactly the rules of System $\mathbf{P}$, which we encountered in similarity analysis of counterfactuals and non-monotonic logic!

Outline

Material Conditional and Probability

Take the degree of belief in an indicative conditional *If* φ*,* ψ to be $P(\varphi \supset \psi)$.

But this does not match how confident we actually are in such conditionals.

Consider a uniformly random card from a standard 52-card deck. Let

$$r := \text{“the card is red”} \qquad k := \text{“the card is a king”}$$

How confident should we be in:

If the card is red, it is a king

Intuitively, this should be given by the conditional probability:

$$P(k \mid r) = \frac{\text{number of red kings}}{\text{number of red cards}} = \frac{2}{26} = \frac{1}{13}$$

If we identify the indicative with the material conditional $r \supset k$, then

$$P(r \supset k) = P(\neg r \vee k) = P(\neg r) + P(k) - P(\neg r \wedge k) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

Adams' Thesis: Conditionals as Assertability Conditions

Adams' thesis: The assertability of an indicative conditional $\varphi \to \psi$ is given by the conditional probability $P(\psi \mid \varphi)$.

We extend the language with a conditional $\to$ satisfying Adams' constraint

$$P(\varphi \to \psi) = P(\psi \mid \varphi)$$

(defined whenever $P(\varphi) > 0$).

If we *do not allow* embedded conditionals (no $\to$ inside φ or ψ), Adams (1975) shows:

Theorem (Adams, Theorem 3.5 and Theorem 4.2)

The consequence relation induced by Adams' probabilistic semantics for (non-embedded) indicative conditionals coincides with system $\mathbf{P}$.

Modus Ponens via $\models_P$

Recall probabilistic entailment:

$$\Gamma \models_P \chi \quad \text{iff} \quad \forall P:\ U_P(\chi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma),$$

where $U_P(\alpha) := P(\neg\alpha)$.

$$\varphi,\ \varphi \to \psi \models_P \psi$$

That is, for every P with $P(\varphi) > 0$,

$$U_P(\psi) \leq U_P(\varphi) + U_P(\varphi \to \psi)$$

Proof (on blackboard).

Paradoxes of Material Implication Avoided

Recall:

$$\varphi_1, \ldots, \varphi_n \models_P \psi \quad \text{iff} \quad \forall P : U_P(\varphi_1) + \cdots + U_P(\varphi_n) \geq U_P(\psi)$$

For a single premise φ:

$$\varphi \models_P \psi \quad \text{iff} \quad \forall P :\ U_P(\varphi) \geq U_P(\psi) \quad \Leftrightarrow \quad \forall P :\ P(\varphi) \leq P(\psi)$$

$P(p \to q) = P(q \mid p)$. We can choose a probability assignment with $P(q) = 0.9$ but $P(q \mid p) = 0.1$. For example:

$$P(p \wedge q) = 0.01,\ P(p \wedge \neg q) = 0.09,\ P(\neg p \wedge q) = 0.89,\ P(\neg p \wedge \neg q) = 0.01.$$

Then $P(q) = 0.9$ and $P(p \to q) = P(q \mid p) = 0.01/0.1 = 0.1$.

So:

$$U_P(q) = 0.1 < 0.9 = U_P(p \to q)$$

showing $q \not\models_P p \to q$.

Finding Invalidities in $\models_P$

To show that

$$\varphi_1, \ldots, \varphi_n \not\models_P \psi$$

we must find a probability function P such that

$$U_P(\varphi_1) + \cdots + U_P(\varphi_n) < U_P(\psi)$$

In practice:

- Draw a Venn diagram for the relevant propositional variables.
- Assign probabilities to the regions.
- Compute $U_P(\cdot)$ and check the inequality.

We use this method in the next examples of valid and invalid inferences.

Some Invalid Inferences

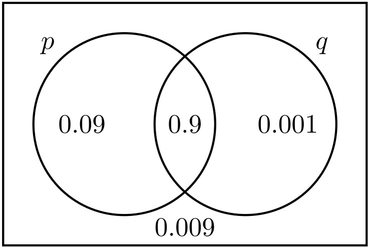
1. $p \supset q \not\models_P p \to q$
2. $p \vee q \not\models_P \neg p \to q$
3. $p \to q \not\models_P \neg q \to \neg p$ (no contraposition)

For (1), take our earlier example with $p =$ "card is red", $q =$ "card is a king." We saw $P(p \supset q) = 7/13$ but $P(p \to q) = P(q \mid p) = 1/13$. So

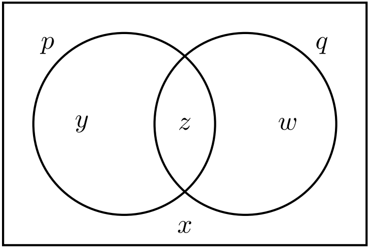
$$U_P(p \supset q) = 6/13 < 12/13 = U_P(p \to q)$$

showing $p \supset q \not\models_P p \to q$.

For (2) and (3), consider:



Some Valid Inferences (I)

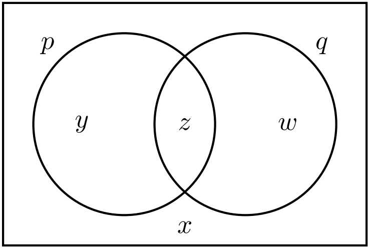


- Let the regions be as in the diagram, so $y = P(p \wedge \neg q)$, $z = P(p \wedge q)$, $w = P(\neg p \wedge q)$, $x = P(\neg p \wedge \neg q)$.
- Then $P(p \supset q) = P(\neg p \vee q) = x + z + w$. Hence $U_P(p \supset q) = 1 - P(p \supset q) = 1 - (x + z + w) = y$.
- $P(p \rightarrow q) = P(q \mid p) = \frac{z}{y+z}$. So $U_P(p \rightarrow q) = 1 - P(p \rightarrow q) = 1 - \frac{z}{y+z} = \frac{y}{y+z}$.
- Since $y \geq 0$ and $0 < y + z \leq 1$, we have $y \leq \dfrac{y}{y+z}$, hence

$$U_P(p \supset q) \leq U_P(p \rightarrow q),$$

so the uncertainty of the conclusion is no greater than that of the premise.

Some Valid Inferences (II) - Modus Ponens



- $p,\ p \rightarrow q \models_P q$.
- From the diagram:
  $U_P(q) = P(\neg q) = x + y$,
  $U_P(p) = P(\neg p) = x + w$,
  $U_P(p \rightarrow q) = 1 - P(q \mid p) = y/(y+z)$
  (with $P(p) = y + z > 0$).
- We must show
  $U_P(q) \leq U_P(p) + U_P(p \rightarrow q)$, i.e.
  $x + y \leq (x + w) + y/(y+z)$.
- This reduces to $y \leq w + y/(y+z)$, which holds because $w \geq 0$ and $y \leq y/(y+z)$ (since $y + z \leq 1$ and $y \geq 0$).
- Hence Modus Ponens is valid for Adams conditionals: $p,\ p \rightarrow q \models_P q$.

Exercise: Axioms

A *probability function* P on a propositional language $\mathcal{L}$ is a function $P : \mathcal{L} \to \mathbb{R}$ such that, for all $\varphi, \psi \in \mathcal{L}$:

1. $P(\varphi) \geq 0$.
2. If $\models \varphi$ (i.e. φ is a tautology), then $P(\varphi) = 1$.
3. If $\models \neg(\varphi \wedge \psi)$ (i.e. φ and ψ are mutually exclusive), then

$$P(\varphi \vee \psi) = P(\varphi) + P(\psi).$$

- Show that if φ and ψ are logically equivalent ($\models \varphi \leftrightarrow \psi$), then $P(\varphi) = P(\psi)$.

4. If $\varphi \models \psi$, then $P(\varphi) \leq P(\psi)$.
5. $P(\varphi) = P(\varphi \wedge \psi) + P(\varphi \wedge \neg\psi)$.

- Show that axiom (3) can be *replaced* by condition (5) More precisely, prove both directions:
  - (a) From (1)-(3) derive and (5).
  - (b) From (1), (2), (5) derive (3).
- Show that replacing (2) and (3) with (4) and (5) respectively does not give you an equivalent characterization.

Exercise: Threshold Consequence and the Lottery Paradox

Work in a purely propositional language (no $\to$ in the object language). Fix a real number n with $0 < n < 1$ and define *threshold consequence*:

$$\Gamma \models^{\geq n} \varphi \quad \text{iff} \quad \forall P : \forall \gamma \in \Gamma,\ P(\gamma) \geq n \ \Rightarrow\ P(\varphi) \geq n$$

- Show that $\Gamma \models^{\geq n} \varphi \Rightarrow \Gamma \models_{CL} \varphi$.
- Show that $\Gamma \models_{CL} \varphi \not\Rightarrow \Gamma \models^{\geq n} \varphi$.

Assume the following *acceptance policy*:

(A1) You *accept* every sentence α such that $P(\alpha) \geq n$.

(A2) Your set of accepted sentences is closed under $\models^{\geq n}$: if Γ is a subset of your accepted sentences and $\Gamma \models^{\geq n} \varphi$, then you also accept φ.

Exercise: Threshold Consequence and the Lottery Paradox

Consider a fair lottery with N tickets (exactly one ticket wins). Let L_i be the sentence “ticket i loses”.

(a) Compute $P(L_i)$ for each i. Show that for N large enough one has $P(L_i) \geq n$ for all i. (Conclude that, by (A1), it is acceptable (on this policy) to accept each L_i separately.)

(b) What is the probability of the conjunction $L_1 \wedge \cdots \wedge L_N$? Explain why this conjunction is in fact *known* to be false in the lottery setup.

Exercise: Threshold Consequence and the Lottery Paradox

(c) Suppose that threshold consequence preserves the classical inference from many premises to their conjunction, i.e.

$$\{L_1, \dots, L_N\} \models^{\geq n} (L_1 \wedge \cdots \wedge L_N).$$

Using (A1) and (A2), show that you are then forced to *accept* the sentence $L_1 \wedge \cdots \wedge L_N$.

(d) Explain why (a)-(c) reproduce the structure of the *lottery paradox*: each L_i is highly probable and acceptable, their conjunction is extremely improbable (indeed impossible), yet closure under consequence forces you to accept it. Which of (A1), (A2), or the expected behaviour of $\models^{\geq n}$ should we give up?

Exercise: Probabilistic consequence P

- Show that the axioms and rules of system $\mathbf{P}$ are sound with respect to probabilistic consequence $\models_P$.
- Show that transitivity fails: $p \to q, q \to r \not\models_P p \to r$

Exercise: Conditionals and material counterparts

Extend the language with an indicative conditional connective $\to$.[2]

For each formula φ (possibly with $\to$), define its *material counterpart* φ^* by:

$$(p)^* = p, \quad (\neg\varphi)^* = \neg\varphi^*, \quad (\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*, \quad (\varphi \to \psi)^* = \varphi^* \supset \psi^*.$$

Let X be a finite set of formulas and $X^* := \{\chi^* : \chi \in X\}$.

(a) Show: If $X \models_P \alpha$ and every formula in $X \cup \{\alpha\}$ is *factual* (i.e. contains no $\to$), then $X^* \models_{CL} \alpha^*$.

(b) Suppose now that α is factual but X may contain conditionals. Show that if $X^* \models_{CL} \alpha^*$, then $X \models_P \alpha$.

[2] The following exercises (from this slide up to *Definability and new variables*) are fairly straightforward. They build on one another and follow basic facts stated in Adams's book, but they are a good way to check your understanding of probabilistic consequence.

Exercise: P-entailment and P-inconsistency

We write $\Gamma \models_P \varphi$ for probabilistic entailment. Say that a finite set Γ is *P-inconsistent* if

$$\Gamma \models_P (\theta \wedge \neg\theta)$$

for some propositional variable θ.

(a) Show that for any finite Γ and formula φ:

$$\Gamma \models_P \varphi \quad \text{iff} \quad \Gamma \cup \{\neg\varphi\} \text{ is P-inconsistent.}$$

(b) Deduce that a finite set Γ P-entails *every* formula iff Γ is P-inconsistent.

Exercise: Proof by cases for $\models_P$

Show the following "proof by cases" principle is valid for $\models_P$:

$$\text{If } \Gamma \cup \{\beta\} \models_P \alpha \text{ and } \Gamma \cup \{\neg\beta\} \models_P \alpha, \text{ then } \Gamma \models_P \alpha.$$

(a) Give an intuitive probabilistic explanation of why this should hold, in terms of uncertainties $U_P(\cdot)$.

(b) Prove it formally from the definition of $\models_P$

Exercise: Minimal p-premises for a conditional

Let φ be a conditional of the form $p \to q$ and let Γ be a finite set of formulas such that:

- Γ contains at least one factual (non-conditional) formula;
- $\Gamma \models_P (p \to q)$;
- no proper subset of Γ P-entails $p \to q$.

Show that: $\Gamma \models_P p$ and $\Gamma \models_P q$.

Exercise: Definability and new variables

Let θ be a propositional variable that does *not* occur in Γ. Let φ be a purely factual formula (no $\to$).

Consider the biconditional $\theta \leftrightarrow \varphi$.

Show that

$$\Gamma \models_P (\theta \leftrightarrow \varphi)$$

iff either

- Γ is P-inconsistent, or
- $\theta \leftrightarrow \varphi$ is classically valid (i.e. $\models_{CL} \theta \leftrightarrow \varphi$).

Explain informally why Γ cannot "force" a non-trivial equivalence between a completely new variable θ and some factual sentence φ, unless Γ itself is already impossible (P-inconsistent).

Outline

Some Reminders

- **Conditional probability.** For $P(B) > 0$:

$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)}$$

- **Chain rule:**

$$P(A \wedge B) = P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$$

- **Law of Total Probability.** If B is any event with $P(B), P(\neg B)$ possibly nonzero, then

$$P(A) = P(A \mid B)\, P(B) + P(A \mid \neg B)\, P(\neg B),$$

  whenever the conditional probabilities are defined.

- **Law of Total Probability (partition).** If $\{B_1, \ldots, B_n\}$ is a partition of Ω with $P(B_i) > 0$, then

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)$$

Conditional Probabilities

Stalnaker's hypothesis (1970): for every conditional $\varphi \to \psi$ (possibly with embedded conditionals),

$$P(\varphi \to \psi) = P(\psi \mid \varphi), \quad \text{with } P(\varphi) > 0$$

Some embedded conditionals are indeed meaningful:

(4) If (the cup broke, if it was dropped), it was fragile.

(5) It is not the case that if I push this button, the light goes on.

Lewis (1976) showed that, together with some plausible principles, this leads to *triviality*: probabilities of conditionals collapse to unconditional probabilities of their consequents.

Lewis Triviality: Ingredients

One form of Lewis triviality (Lewis 1979, 1989) result uses these assumptions:

1. **Adams thesis:** $P(\varphi \to \psi) = P(\psi|\varphi)$ for all relevant φ, ψ.
2. **Import-export:** $\varphi \to (\psi \to \chi) \equiv (\varphi \wedge \psi) \to \chi$.
3. **Stalnaker hypothesis:** $\varphi \to \psi$ is a proposition (an element of the same algebra as ordinary sentences), so it can itself be conditionalized on.

Lewis shows that these together imply for φ, ψ:

$$P(\psi \mid \varphi) = P(\psi)$$

i.e. conditioning on φ never changes the probability of ψ, which is absurd in general.[3]

[3] Lewis notes that this is harmless if P never gives positive probability to more than two incompatible propositions. This means that P has at most four distinct probability values. Can you see why? Hint: think of a 1-circle/2-cell Venn partition.

Triviality Result: Schematic Derivation

1. Import-export: $\alpha \to (\varphi \to \psi) \equiv (\alpha \land \varphi) \to \psi$
2. By Adams thesis: $P(\varphi \to \psi \mid \alpha) = P(\psi \mid \alpha \land \varphi)$ This holds for all α.
3. Apply the Law of Total Probability to $\varphi \to \psi$ with respect to ψ:

$$P(\varphi \to \psi) = P(\varphi \to \psi \mid \psi)\, P(\psi) + P(\varphi \to \psi \mid \neg\psi)\, P(\neg\psi)$$

4. Using (2) with $\alpha = \psi$ and $\alpha = \neg\psi$:

$$P(\varphi \to \psi \mid \psi) = P(\psi \mid \varphi \land \psi)$$
$$P(\varphi \to \psi \mid \neg\psi) = P(\psi \mid \varphi \land \neg\psi)$$

5. So

$$P(\varphi \to \psi) = P(\psi \mid \varphi \land \psi)\, P(\psi) + P(\psi \mid \varphi \land \neg\psi)\, P(\neg\psi)$$

6. Adams thesis also gives $P(\varphi \to \psi) = P(\psi \mid \varphi)$, hence

$$P(\psi \mid \varphi) = P(\psi \mid \varphi \land \psi)\, P(\psi) + P(\psi \mid \varphi \land \neg\psi)\, P(\neg\psi)$$

7. But $P(\psi \mid \varphi \land \psi) = 1$ and $P(\psi \mid \varphi \land \neg\psi) = 0$, so

$$P(\psi \mid \varphi) = P(\psi)$$

What to Give Up?

Something has to go. Options:

- **Propositional status**: deny that $\varphi \to \psi$ always denotes an ordinary proposition (Edgington 1995)
- **Adams's simple thesis**: adopt more complex probability-based accounts where $P(\varphi \to \psi)$ is not just $P(\psi \mid \varphi)$. (Douven 2016; Berto & Özgün 2021)
- **Classical probability laws**: modify the underlying probability theory (e.g. Ciardelli and Odmussen 2024)
- **Non-probabilistic** treatment of indicative conditionals (Angelika Kratzer, Anthony Gillies).

Outline

Priors, Posteriors, and Bayes' Rule

- We often compare hypotheses H in the light of new evidence E.
- **Prior $P_0(H)$:** how plausible H is *before* learning E.
- **Posterior $P_1(H)$:** how plausible H is *after* learning E.

We update our belief in H by combining
(1) how plausible H already was, with
(2) how well H predicts the new evidence.

$$P_1(H) \;=\; P_0(H \mid E) \;=\; \frac{P_0(E \mid H)\, P_0(H)}{P_0(E)}$$

- **Prior** $P_0(H)$
- **Likelihood** $P_0(E \mid H)$: how unsurprising E would be if H were true.
- **Normalization** $P_0(E)$: rescales so posteriors add up to 1 across competing hypotheses.

The Epistemic Past Hypothesis (Adams, Ch. 4)

A counterfactual can function as an *epistemic past tense*: after new evidence is learned, we assess $\varphi \rightsquigarrow \psi$ by looking to the *prior* assertability of the corresponding indicative.

$$P_1(\varphi \rightsquigarrow \psi) \;=\; P_0(\psi \mid \varphi) \qquad \text{(prior conditional probability hypothesis).}$$

Let C be "the urn is type C" and Y be "the ball is yellow".

$$P_0(C) = P_0(\neg C) = \tfrac{1}{2}, \quad P_0(\neg Y \mid C) = 0.01, \quad P_0(\neg Y \mid \neg C) = 0.80.$$

You draw a non-yellow ball. Then:

$$\frac{P_1(\neg C)}{P_1(C)} = \frac{P_0(\neg C)}{P_0(C)} \cdot \frac{P_0(\neg Y \mid \neg C)}{P_0(\neg Y \mid C)} = 80.$$

It is natural here to hear $P_0(\neg Y \mid C)$ as an *inverse-prior* probability that matches a counterfactual gloss: *if the urn were C, a yellow ball would not have been drawn.*

Generalizing: The Hypothetical Epistemic Past

- The Epistemic Past idea: a counterfactual can reflect what the corresponding indicative *would have been* assertible *earlier*.
- But sometimes there is no *actual* earlier standpoint from which anyone could reasonably assert the indicative.
- In such cases, the counterfactual looks to a *hypothetical* epistemic past.

If Napoleon had been kept under stricter guard on Elba, he would not have escaped, and Waterloo would never have happened.

- No one plausibly occupied the relevant *actual* prior position.
- Yet one could occupy a *counterfactual* prior position where the corresponding indicative would be assertible.

The Button-and-Light Counterexample

Two buttons, A and B. The light L goes on iff *exactly one* button was pushed:

$$L \equiv (A \wedge \neg B) \ \vee \ (\neg A \wedge B).$$

Priors:

$$P_0(A) = \tfrac{1}{1000}, \qquad P_0(B) = \tfrac{1}{1{,}000{,}000}.$$

Assuming that $P_0(\neg A \mid B) = P_0(\neg A)$ and $P_0(\neg B \mid A) = P_0(\neg B)$:

$$P_0(L \mid B) = P_0(\neg A) = 0.999, \qquad P_0(L \mid A) = P_0(\neg B) = 0.999999.$$

$$\frac{P_0(B)}{P_0(A)} = 0.001.$$

$$P_0(\neg L \mid B) = P_0(A) = 0.001.$$

So the simple epistemic-past identification would make $B \rightsquigarrow \neg L$ *very* unlikely.

The Button-and-Light Counterexample

You learn that the light is on.

Since $A \wedge L \equiv A \wedge \neg B$ and $B \wedge L \equiv B \wedge \neg A$,

$$\frac{P_1(B)}{P_1(A)} = \frac{P_0(B \wedge \neg A)}{P_0(A \wedge \neg B)}.$$

This yields:

$$\frac{P_1(B)}{P_1(A)} = \frac{P_0(B)}{P_0(A)} \cdot \frac{P_0(\neg A)}{P_0(\neg B)} = 0.001 \cdot \frac{0.999}{0.999999} = \frac{999}{999999} = \frac{1}{1001}.$$

So, upon observing L, it is about 1001 times likelier that A was pushed than B.

We are inclined to affirm $B \rightsquigarrow \neg L$, but the simple epistemic-past hypothesis would tie this to $P_0(\neg L \mid B) = 0.001$. So the epistemic past identification breaks.

Adams's Repair Attempt: a two-factor model

Assume mutually exclusive/exhaustive states $S_1, \ldots, S_n$ (causally independent of B) that, together with B, determine $\neg L$. Then:

$$P(B \rightsquigarrow \neg L) = \sum_{i=1}^{n} P_1(S_i)\, P_0(B \wedge S_i \rightsquigarrow \neg L).$$

So after what you have learned, you evaluate 'If B, then not L' by averaging over the different background possibilities, weighted by how likely they now seem.

Take $S_1 = A$, $S_2 = \neg A$.

$$P_0(B \wedge A \rightsquigarrow \neg L) = 1, \qquad P_0(B \wedge \neg A \rightsquigarrow \neg L) = 0,$$

so

$$P(B \rightsquigarrow \neg L) = P_1(A).$$

And indeed

$$P_1(A) = P_0(A \mid L) \approx 1,$$

matching the strong post-observation counterfactual evaluation.